# Gain Margins and Phase Margins for Control Systems with Adjustable Parameters

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**This paper presents a method for finding the gain margins and phase margins of control systems with adjustable parameters. The considered systems are first modified by adding an analytical gain-phase margin tester. Then, the characteristic equations are formulated. Finally, the stability equations are used to find the boundaries of constant gain margins and phase margins. The main advantage of the proposed method is to obtain complete information about the effects of adjustable parameters on gain margins and phase margins. As an example, the method is applied to a pitch-rate control system for a re-entry vehicle, and comparisons with the results in current literature are made.**

## I. Introduction

FOR analysis and design of practical control systems, gain margin and phase margin are the two important specifications. The frequency domain approach, based on the works of Nyquist, Bode, and Nichols, permits a designer to find the gain and phase margins in a simple manner. However, these frequency domain approaches are not suitable for systems with two or more adjustable parameters. On the other hand, the Vishnegradskii diagram,[1] the parameter plane method,[2,3] and the stability-equation method[4] are all useful for plotting the stability boundaries and finding the effects of parameter variations, but no result related to phase margin and gain margin has been given.

The main purpose of this paper is to extend the aforementioned methods to provide information for plotting the boundaries of constant gain margin and phase margin in a parameter plane or a parameter space. Then, the effects of adjustable parameters can be clearly defined. In addition, the phase crossover frequency and the gain crossover frequency can be obtained directly.

## II. Basic Approach

Consider the system shown in Fig. 1. The open-loop transfer function is

$$G(s) = N(s)/D(s) \tag{1}$$

Let $s = j\omega$; then, Eq. (1) becomes

$$G(j\omega) = N(j\omega)/D(j\omega) \tag{2}$$

which can be written as

$$G(j\omega) = \mathrm{Re}[G(j\omega)] + j\,\mathrm{Im}[G(j\omega)] \tag{3}$$

where $\mathrm{Re}[G(j\omega)]$ and $\mathrm{Im}[G(j\omega)]$ are the real and imaginary parts of $G(j\omega)$, respectively. Eq. (2) can be expressed in terms of its magnitude and phase, such as

$$G(j\omega) = |G(j\omega)|e^{j\phi} \tag{4}$$

where

$$|G(j\omega)| = \sqrt{\mathrm{Re}[G(j\omega)]^2 + \mathrm{Im}[G(j\omega)]^2} \tag{5}$$

and

$$\phi = \underline{/G(j\omega)} = \tan^{-1}\left\{\mathrm{Im}[G(j\omega)]\big/\mathrm{Re}[G(j\omega)]\right\} \tag{6}$$

From Eqs. (2) and (4), one has

$$D(j\omega)|G(j\omega)|e^{j\phi} - N(j\omega) = 0 \tag{7}$$

i.e.,

$$D(j\omega) - (1/|G(j\omega)|e^{j\phi})N(j\omega) = 0 \tag{8}$$

Define

$$1/|G(j\omega)| = A \tag{9}$$

$$\phi + 180\ \mathrm{deg} = \Theta \tag{10}$$

Then, Eq. (8) becomes

$$D(j\omega) + Ae^{-j\Theta}N(j\omega) = 0 \tag{11}$$

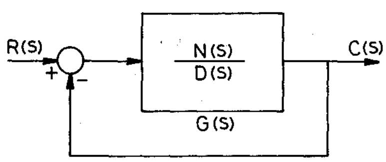


**Fig. 1 A basic block diagram of a control system.**

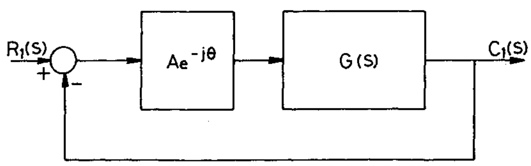


**Fig. 2 A control system with a gain-phase margin tester.**

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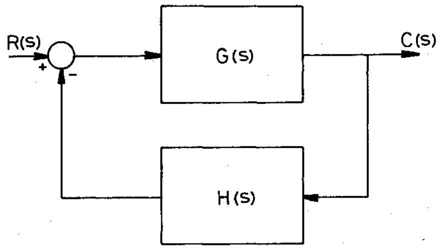



**Fig. 3 Block diagram of a general feedback system.**

Note that $A$ is the gain margin of the system if $\Theta = 0$, and that $\Theta$ is the phase margin of the system if $A = 1$. This can be checked by use of a Nyquist plot. The physical meaning of Eq. (11) is that the gain margin and the phase margin of a system can be determined by the gain-phase margin tester $Ae^{-j\Theta}$, which can be considered an additional block, as shown in Fig. 2. Equation (11) can also be considered as

$$F(j\omega) = 1 + Ae^{-j\Theta}G(j\omega) = 0 \tag{12}$$

which indicates that the gain margin and the phase margin of the system can be determined from the characteristic equation of the system with a gain-phase margin tester. Note that for a control system with block diagram, as shown in Fig. 3, the loop transfer function $G(j\omega)H(j\omega)$ should be used to replace $G(j\omega)$ in Eq. (12).[5]

The gain-phase margin tester can also be expressed as

$$Ae^{-j\Theta} = A\cos\Theta - jA\sin\Theta = X - jY \tag{13}$$

where

$$X = A\cos\Theta \tag{14}$$

$$Y = A\sin\Theta \tag{15}$$

Then, the characteristic equation can be written as

$$F(j\omega) = D(j\omega) + (X - jY)N(j\omega) = 0 \tag{16}$$

which can be expressed in terms of a real and an imaginary part; that is,

$$F(j\omega) = Fr(X,Y,\omega) + jFi(X,Y,\omega) = 0 \tag{17}$$

Eq. (17) gives

$$Fr(X,Y,\omega) = 0 \tag{18}$$

$$Fi(X,Y,\omega) = 0 \tag{19}$$

which are the two stability equations.[4] Consider $X$ and $Y$ as parameters; then, one has

$$Fr(X,Y,\omega) = X\cdot B_1 + Y\cdot C_1 + D_1 = 0 \tag{20}$$

$$Fi(X,Y,\omega) = X\cdot B_2 + Y\cdot C_2 + D_2 = 0 \tag{21}$$

where $B_1$, $B_2$, $C_1$, $C_2$, $D_1$, and $D_2$ are functions of $\omega$. Solving Eqs. (20) and (21) simultaneously for $X$ and $Y$, one has

$$X = \frac{C_1\cdot D_2 - C_2\cdot D_1}{\Delta} \tag{22}$$

$$Y = \frac{D_1\cdot B_2 - D_2\cdot B_1}{\Delta} \tag{23}$$

where

$$\Delta = B_1\cdot C_2 - B_2\cdot C_1 \tag{24}$$

By letting $\omega$ vary from zero to $\infty$, a locus for common roots of Eqs. (22) and (23) can be plotted in the $X$ vs $Y$ plane. This locus contains the stability boundary of the system, which has a gain-phase margin tester. The intersection point of this locus with the $X$ axis gives the gain margin of the system, and the corresponding value of $\omega$ is the phase crossover frequency. Similarly, the intersection point of this locus with the unit circle gives the value of the phase margin, and the corresponding value of $\omega$ is the gain crossover frequency.

If the system has variable and/or adjustable parameters, Eq. (12) can be written as

$$F(j\omega) = F(\alpha,\beta,\gamma,\ldots,A,\Theta,j\omega) = 0 \tag{25}$$

where $\alpha,\beta,\gamma,\ldots$ are parameters. Doing some algebraic manipulations and decomposing the characteristic equation into two stability equations, one has

$$Fr(\alpha,\beta,\gamma,\ldots,A,\Theta,\omega) = 0 \tag{26}$$

and

$$Fi(\alpha,\beta,\gamma,\ldots,A,\Theta,\omega) = 0 \tag{27}$$

Assume that Eqs. (26) and (27) are linear functions of $\alpha$ and $\beta$. (Note that this restriction can be lifted, as stated in Refs. 2 and 3.) Then, one has

$$Fr(\alpha,\beta,\gamma,\ldots,A,\Theta,\omega) = \alpha\cdot B_1 + \beta\cdot C_1 + D_1 = 0 \tag{28}$$

$$Fi(\alpha,\beta,\gamma,\ldots,A,\Theta,\omega) = \alpha\cdot B_2 + \beta\cdot C_2 + D_2 = 0 \tag{29}$$

where $B_1$, $B_2$, $C_1$, $C_2$, $D_1$, and $D_2$ are functions of $\gamma\ldots,A,\Theta,\omega$. Solving Eqs. (28) and (29) for $\alpha$ and $\beta$, one has

$$\alpha = \frac{C_1\cdot D_2 - C_2\cdot D_1}{\Delta} \tag{30}$$

$$\beta = \frac{D_1\cdot B_2 - D_2\cdot B_1}{\Delta} \tag{31}$$

where

$$\Delta = B_1\cdot C_2 - B_2\cdot C_1 \tag{32}$$

Let $A = 1$ and $\Theta = 0$, and set $\gamma\ldots$ equal to constants; then, for various values of $\omega$, a locus that contains the stability boundary of the system without the gain-phase margin tester can be plotted in the $\alpha$ vs $\beta$ plane. This locus can be considered the Nyquist plot of the system passing through the critical point $(-1,j0)$, which indicates that the numerator of Eq. (12) has at least one pair of roots on the imaginary axis of the $s$-plane. If $A$ is assumed equal to a constant value and $\Theta = 0$, the locus in the $\alpha$ vs $\beta$ plane is a boundary of constant gain margin. On the other hand, if $A = 1$ and $\Theta$ is assumed equal to a constant value, then the locus is a boundary of constant phase margin. The corresponding values of $\omega$ on the constant gain margin boundary and the constant phase margin boundary are the phase crossover frequency and gain crossover frequency, respectively.

For each specified value of $\gamma$, the stability boundary, the constant gain margin boundary and the constant phase margin boundary can be found. For several values of $\gamma$, a subspace can be found in a three-dimensional parameter space, using $\gamma$ as the third axis.[6-8]

## III. Examples

**Example 1**

Consider the system shown in Fig. 1, if the open-loop transfer function is

$$G(s) = 5000/s(s^2 + 15s + 50) \tag{33}$$

After the gain-phase margin tester is added into the system, as shown in Fig. 2, and $s = j\omega$, then the characteristic equation of the system is

$$F(j\omega) = D(j\omega) + (X - jY)N(j\omega)$$
$$= (-j\omega^3 - 15\omega^2 + j50\omega) + (X - jY)5000 = 0 \quad (34)$$

which gives the two stability equations

$$Fr(X,Y,\omega) = X \cdot B_1 + Y \cdot C_1 + D_1$$
$$= X[5000] + Y[0] + [-15\omega^2] = 0 \quad (35)$$

$$Fi(X,Y,\omega) = X \cdot B_2 + Y \cdot C_2 + D_2$$
$$= X[0] + Y[-5000] + [-\omega^3 + 50\omega] = 0 \quad (36)$$

where $B_1 = 5000$, $C_1 = 0$, $D_1 = -15\omega^2$, $B_2 = 0$, $C_2 = -5000$, and $D_2 = -\omega^3 + 50\omega$. Using Eqs. (22–24), and letting $\omega$ vary from zero to $\infty$, a locus is plotted, as shown in Fig. 4, which indicates the following. The gain margin ($A$) of the system is 0.15 ($-16.48$ dB) and the phase crossover frequency is at 7.07 rad/s, and the phase margin ($\Theta$) is $-40.5°$ and the gain crossover frequency is at 16 rad/s. The main purpose of this example is to indicate that by use of the proposed method the gain margin and phase margin can be obtained directly in the $Y$ vs $X$ plane. This $Y$ vs $X$ plane is different from the original Nyquist diagram, but the same result can be obtained by use of the Nyquist plot. The usefulness of the proposed method will be illustrated by the following examples.

**Example 2**

Consider the system shown in Fig. 3, where

$$G(s) = 1000(s + \alpha)/s(s^2 + 15s + 50) \quad (37)$$

and

$$H(s) = 5/(s + \beta) \quad (38)$$

The loop transfer function $G_t(s)$ is

$$G_t(s) = \frac{5000(s + \alpha)}{s(s + \beta)(s^2 + 15s + 50)} \quad (39)$$

where $\alpha$ and $\beta$ are two parameters. Inserting the gain-phase margin tester into the system and setting $s = j\omega$, the characteristic equation is

$$F(j\omega) = 1 + Ae^{-j\Theta}G_t(j\omega)$$
$$= 1 + Ae^{-j\Theta}\frac{5000(j\omega + \alpha)}{j\omega(j\omega + \beta)(-\omega^2 + j15\omega + 50)} = 0 \quad (40)$$

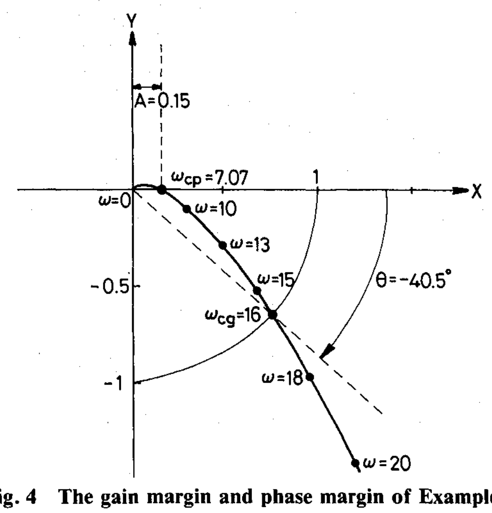



**Fig. 4 The gain margin and phase margin of Example 1.**

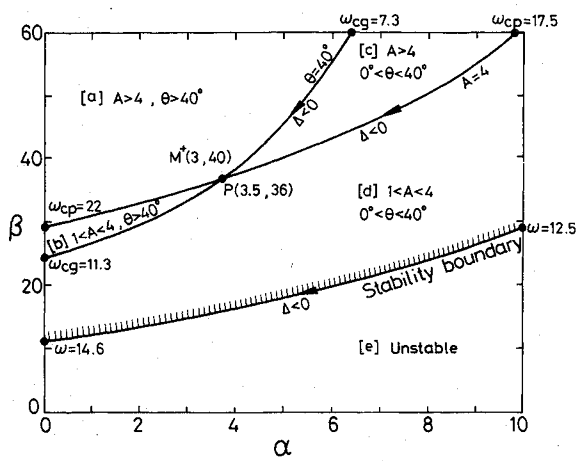



**Fig. 5 The boundaries of constant gain margin and constant phase margin in the $\alpha$ vs $\beta$ plane.**

After some algebraic manipulations, the two stability equations can be expressed as

$$Fr(\alpha,\beta,A,\Theta,\omega) = \alpha \cdot B_1 + \beta \cdot C_1 + D_1 = \alpha \cdot [5000A\ \cos\Theta]$$
$$+ \beta \cdot [-15\omega^2] + [\omega^4 - 50\omega^2 + 5000A\ \sin\Theta \cdot \omega] \quad (41)$$

$$Fi(\alpha,\beta,A,\Theta,\omega) = \alpha \cdot B_2 + \beta \cdot C_2 + D_2 = \alpha \cdot [-5000A\ \sin\Theta]$$
$$+ \beta \cdot [-\omega^3 + 50\omega] + [-15\omega^3 + 5000A\ \cos\Theta \cdot \omega] \quad (42)$$

Assume $A = 4$, $\Theta = 0$ and $\Theta = 40$ deg, $A = 1$. Using Eqs. (30–32), and letting $\omega$ vary from zero to $\infty$, a boundary of constant gain margin ($A = 4$) and a boundary of constant phase margin ($\Theta = 40$ deg) are plotted in the $\alpha$ vs $\beta$ plane, as shown in Fig. 5, where the stability boundary of the system without the gain-phase margin tester is also shown. Figure 5 indicates that there are five regions in the $\alpha$ vs $\beta$ plane, and each region has its specific gain margin and phase margin. If $\alpha$ and $\beta$ are adjusted to point $P(\alpha = 3.5, \beta = 36)$, the system will have a gain margin at $A = 4$ and a phase margin at $\Theta = 40$ deg. The corresponding phase crossover frequency $\omega_{cp}$ and gain crossover frequency $\omega_{cg}$ are at 20.7 rad/s and 9.3 rad/s, respectively. If $\alpha$ and $\beta$ are adjusted to point $M(\alpha = 3, \beta = 40)$ in region [$a$], it can be predicted that the system will have $A > 4$ and $\Theta > 40$ deg. The results in this example have also been checked by use of the method in Example 1 and by Nyquist plots.

Note that the main purpose of this example is to show that the proposed method can be applied to a system with a transfer function in its feedback path; thus, the capabilities of effector and other units are not considered.

**Example 3**

Assume that the system shown in Fig. 1 has an unstable open-loop transfer function, such as

$$G(s) = \frac{N(s)}{D(s)} = \frac{2(s + \alpha)(s + 3)}{s(s + \beta)(s - 1)} \quad (43)$$

The characteristic equation of the system with the gain-phase margin tester is

$$F(j\omega) = j\omega(j\omega + \beta)(j\omega - 1) + 2Ae^{-j\Theta}(j\omega + \alpha)(j\omega + 3) = 0 \quad (44)$$

The stability equations are

$$Fr(\alpha,\beta,A,\Theta,\omega) = \alpha \cdot B_1 + \beta \cdot C_1 + D_1$$
$$= \alpha \cdot [6A\ \cos\Theta + 2A\ \sin\Theta \cdot \omega] + \beta \cdot [-\omega^2]$$
$$+ [\omega^2 - 2A\ \cos\Theta \cdot \omega^2 + 6A\ \sin\Theta \cdot \omega] \quad (45)$$

$$Fi(\alpha,\beta,A,\Theta,\omega) = \alpha \cdot B_2 + \beta \cdot C_2 + D_2$$
$$= \alpha \cdot [2A\ \cos\Theta \cdot \omega - 6A\ \sin\Theta] + \beta \cdot [-\omega]$$
$$+ [-\omega^3 + 6A\ \cos\Theta \cdot \omega + 2A\ \sin\Theta \cdot \omega^2] \quad (46)$$

By use of the same approach, the boundaries of constant gain margin ($A = 0.3$) and constant phase margin ($\Theta = 30$ deg) are plotted, as shown in Fig. 6. If $\alpha$ and $\beta$ are adjusted to point $P(\alpha = 17.78, \beta = 8.53)$, the system will have a gain margin at $A = 0.3$ and a phase margin at $\Theta = 30$ deg. If $\alpha$ and $\beta$ are adjusted to point $M_1(\alpha = 50, \beta = 20)$ or $M_2(\alpha = 0.2, \beta = 1)$, the system will have $A < 0.3$ and $\Theta > 30$ deg. Note that the region for gain margin $A < 1$ is inside the stable region, because the transfer function $G(s)$ has a pole in the right-side of the $s$-plane.

Examples 2 and 3 show that the proposed method can be applied to various kinds of control systems to obtain information about the effects on gain margin and phase margin due to variable and/or adjustable parameters. One does not need to care about the signs of the gain margin (in decibels) and the phase margin, because one generally finds the stable region first and then finds the boundaries of constant gain margin and phase margin inside the stable region. For example, in Example 2 the positive gain margin (in decibels) and positive phase margin represent the relative stability of the system, whereas in Example 3 the negative gain margin (in decibels) and the positive phase margin represent the relative stability of the system.

**Example 4**

The block diagram of an aerodynamically unstable maneuverable re-entry vehicle system is shown in Fig. 7.[9] Based on this system configuration, the loop transfer function is

$$G(s) = \frac{Kq \cdot Q(s)}{Kp \cdot P(s) + T(s)} \tag{47}$$

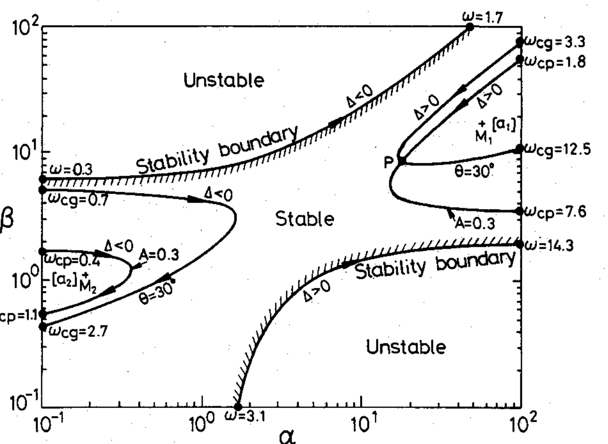



**Fig. 6 The boundaries of constant gain margins and constant phase margins of Example 3.**

where

$$P(s) = s(s \pm 24.8)(s + 471. \pm j816)(s \pm j5030) \tag{48}$$

$$Q(s) = (s + 30)(s + 3.6)(s \pm j910) \tag{49}$$

$$T(s) = s(s + 30)(s \pm 24.8)(s + 448. \pm j523) \times (s \pm j3030)(s + 471. \pm j816) \tag{50}$$

$$Kp = 6.2232 \cdot 10^5 \alpha \tag{51a}$$

$$Kq = 4.3173 \cdot 10^{15} \beta \tag{51b}$$

Note that

$$(s \pm 24.8) = (s + 24.8)(s - 24.8) \quad \text{etc.}$$

The servo gain $\alpha$ and the rate gain $\beta$ are adjustable parameters. The characteristic equation of the system with the gain-phase margin tester is

$$F(j\omega) = 1 + Ae^{-j\Theta} \frac{Kq \cdot Q(j\omega)}{Kp \cdot P(j\omega) + T(j\omega)} \tag{52}$$

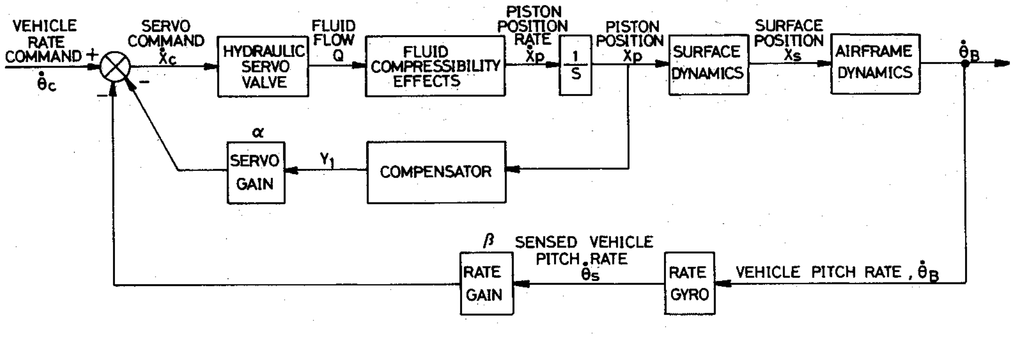
One can take $Kp$ and $Kq$ as parameters, because they are proportional to the servo gain $\alpha$ and the rate gain $\beta$. After some algebraic manipulations, one finds the two stability equations

$$Fr(Kp, Kq, A, \Theta, \omega) = Kp \cdot [Pr] + Kq \times [Qr \cdot A \cos\Theta + Qi \cdot A \sin\Theta] + Tr \tag{53}$$

$$Fi(Kp, Kq, A, \Theta, \omega) = Kp \cdot [Pi] + Kq \times [Qi \cdot A \cos\Theta + Qr \cdot A \sin\Theta] + Ti \tag{54}$$

where $Pr$, $Qr$, and $Tr$ are the real parts of $P(j\omega)$, $Q(j\omega)$, and $T(j\omega)$, respectively; $Pi$, $Qi$, and $Ti$ are the imaginary parts of $P(j\omega)$, $Q(j\omega)$, and $T(j\omega)$, respectively.

By applying the same method and transferring $Kp$ and $Kq$ to the servo gain ($\alpha$) and the rate gain ($\beta$), one may plot the stability boundary, the boundaries of constant gain margins ($A = 2$, $A = 3$, $A = 1/2$, $A = 1/3$), and constant phase margins ($\Theta = 15$, 30, and 45 deg) in the $\alpha$ vs $\beta$ plane, as shown in Fig. 8. It can be seen that the gain margins for $A > 1$ and $A < 1$ both appear in the stable region, which means that the system has two gain margins (one is greater than 1 and the other is smaller than 1). Although there are many regions in




$$\frac{Q}{\dot{X}_c} = \frac{(7.7)(700)^2}{S^2 + 2(0.65)(700)S + (700)^2} \qquad \frac{X_s}{X_p} = \frac{(5030)^2}{S^2 + (5030)^2}$$

$$\frac{\dot{X}_p}{Q} = \frac{(1/2.2)(3030)^2(S^2 + (5030)^2)}{(5030)^2(S^2 + (3030)^2)} \qquad \frac{\dot{\theta}_B}{X_s} = \frac{(3.09 \times 10^{-4})(S + 3.63)(S^2 + (910)^2)}{S^2 - (24.8)^2}$$

$$\frac{Y_1}{X_p} = \frac{S}{S + 30} \qquad \frac{\dot{\theta}_s}{\dot{\theta}_B} = \frac{(942)^2}{S^2 + (2)(.5)(942)S + (942)^2}$$

**Fig. 7 Block diagram of a control system for a re-entry vehicle.**

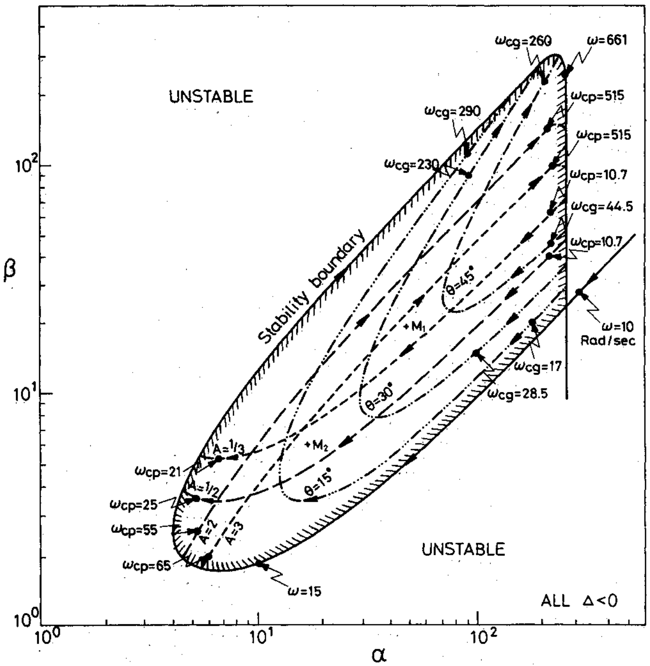



**Fig. 8 The boundaries of constant gain margins and phase margins of Example 4.**

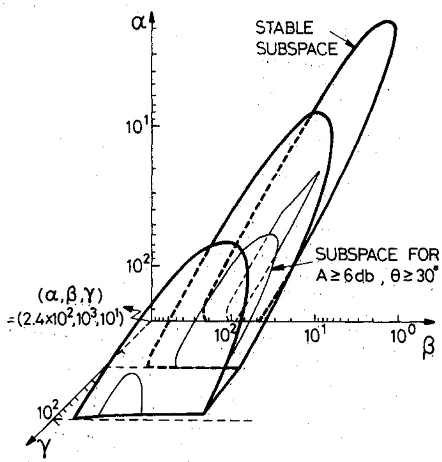



**Fig. 9 Parameter-space plots for Example 4.**

the plane, each region has its specified characteristics on gain margin and phase margin. For example, the region marked by $M_1$ will give gain margin $A > 3$ and $A < 1/3$, and phase margin 30 deg $< \Theta <$ 45 deg; the region marked by $M_2$ will give $A > 3$, $1/3 < A < 1/2$ and 15 deg $< \Theta <$ 30 deg.

Figure 9 is a three-dimensional parameter-space, where the third axis ($\gamma$) is the pole of the compensator in Fig. 7. There are two subspaces in Fig. 9: the outer one represents the stable subspace; the inner one is the subspace that satisfies gain margin and phase margin larger than 6 dB and 30 deg, respectively.

The system in this example has been considered in Ref. 9. It should be noted that the phase margins and gain margins obtained in our paper are based on the loop transfer function, as defined in Eq. (47); therefore, all the results can be checked by the commonly used Nyquist plots. But, in Ref. 9 the gain margins and phase margins are based on the characteristic equation, not on the loop transfer function, as defined in Eq. (47); therefore, the results obtained in Ref. 9 are different from our results shown in Fig. 8. Thus, it is worthwhile to note that only the characteristic equation [Eq. (12)] based on the loop transfer with a gain-phase margin tester as proposed in this paper can be used to find the gain margin and phase margin. In other words, the boundaries of constant gain margin and phase margin obtained in Ref. 9 are incorrect, except that the stability boundary is correct.

Note that the presented technique can be used to augment other "parameter plane (space)" techniques[2,3,6,7] to yield pertinent additional information.

## IV. Conclusions

A method for finding the gain margins and phase margins of control systems with adjustable parameters has been proposed. The main advantage of the proposed method is that the relations among gain margins, phase margins, and adjustable parameters can be defined completely and easily. Since the method can be used to express the stability boundary and the boundaries of constant gain margin and phase margin in a parameter-plane or a parameter-space, it is a useful tool for analysis and design.

In addition, since the analyses are based on two stability equations, which are amenable to digital computer computation, the proposed method has the potential for analysis and design of very complicated control systems.

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